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QUANTIFYING ASSET PRICE VOLATILITY WITH FRACTIONAL BROWNIAN MOTION

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INTRODUCTION

Introduction This research focuses on the application of stochastic analysis in mathematical finance, where the price of assets is modeled as a realization of a random process. Specifically, we employ fractional Brownian motion as a driver for the dynamic model of price, which is neither a semimartingale nor Markovian. This approach captures the "market memory" effect, which is overlooked by the conventional methods based on classical Brownian motion. To analyze the volatility of multiple assets, we employ statistical methods based on p-variations, which take into account volatility changes over time. The efficacy of the p-variation method for volatility estimation is demonstrated through empirical analysis of financial data from 2020. In particular, we show that the method accurately captures the volatility explosion that occurred during the COVID-19 pandemic.

Theoretical background

Brownian diffusion approaches such as Heston and CEV models do not generate implied volatility surfaces that align well with the actual data. One of the main reasons for that outlined in the literature is the fact that Markovian models fail to capture the market memory effect, i.e. non-trivial dependence of future asset prices on the entire history of the latter. A widely used technique for modeling markets with memory is to substitute the standard Brownian motion in stochastic models with the fractional Brownian motion $B^H = \{B^H(t), t \geq 0\}$ which is, in a certain sense, the only Gaussian process enjoying self-similarity together with the stationarity of increments.

Model

In this study, we assume that the asset price

$$S = \{S(t), t \geq 0\}$$

is given by,

$$S(t) = e^{X(t)},$$

where $X = \{X(t), t \geq 0\}$ is a fractional diffusion of the form

$$X(t) = X(0) + \int_0^t \beta(s)ds + \int_0^t \sigma(s)dB^H(s),$$

where B^H denotes a fractional Brownian motion with unknown Hurst index $H \in (0,1)$, i.e. a centered Gaussian process with covariance function

$$E[B^H(s)B^H(t)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$$

Our goal is to estimate the unknown Hurst index H and the volatility $\sigma = \sigma(t)$ and, in order to do that, we will employ the technique of p-variations as described in [5].

Estimation method

Assume that the log-price X is observed on a uniform partition $\{0 = t_0 < t_1 < \ldots < t_n = T\}$ of an interval $[0,T]$, $t_k = \frac{k}{n}$. Denote

$$V^n(X)_t := \sum_{k=1}^{[nt]-1} |X(t_k) - X(t_{k-1})|^2$$

$$V^{2n}(X)_t := \sum_{l=1}^{\left[\frac{nt}{2}\right]-1} |X(t_{2l}) - X(t_{2l-2})|^2$$

- The estimate $\hat{H} := \frac{1}{2} - \frac{V^{2n}(X)_T - V^n(X)_T}{2log2}$ a (weakly) consistent estimator of the Hurst index.
- Given the Hurst index, the stochastic process $\hat{I}(t) := \frac{1}{n^{1-2H}} V^n(X)_t$ converges uniformly in probability on compact sets to the stochastic process $I(t) := \int_0^t \sigma^2(s)\, ds$ i.e. $\hat{I}(t)$ is a uniform (w.r.t. t) weakly consistent estimator of the integrated squared volatility.
- Now fix a natural number $m << n$ and denote $\pi_{n,m}(t)$ the subset of partition containing only those points that are belong to $[t - m/n, t + m/n)$. Then, as the point estimator of the volatility function $\sigma(t)$, one can take

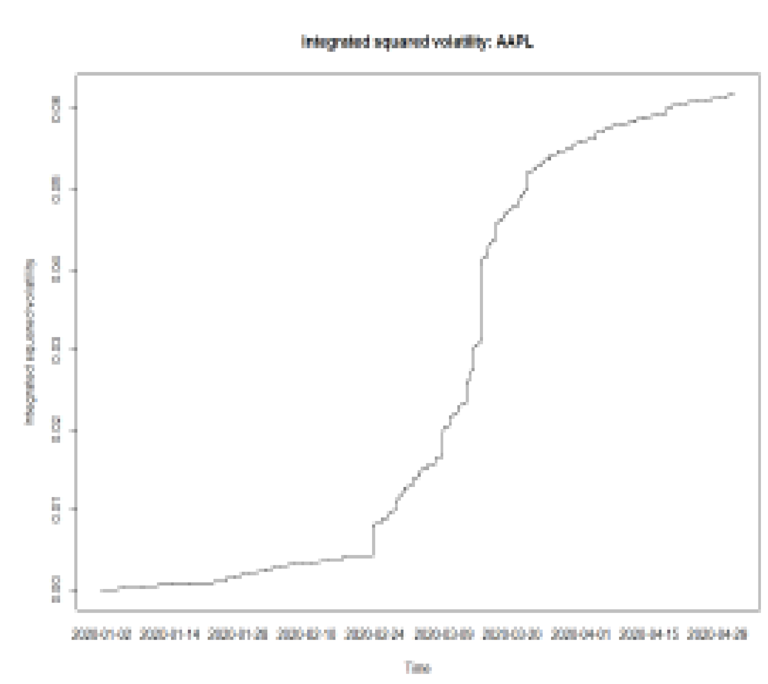
$$\hat{\sigma}(t) := \frac{n}{2m} \sum_{k: t_k \in \pi_{n,m(t)}} (\hat{I}\left(t + \frac{1}{n}\right) - \hat{I}(t)$$

Data

To showcase the efficacy of our estimation methodology, we employed it to analyze high-frequency (minute-by-minute) data obtained from the global financial markets. The aforementioned estimations were conducted on diverse financial instruments, such as ordinary stocks of Apple (31837 observations), Dow Jones industrial average index (33701 observations), ordinary stocks of Amazon (24330 observations), and the exchange rate of BitCoin cryptocurrency to USD (165299 observations), over the period of January 1 to May 1, 2020.

Instrument	AAPL	AMZN	BTC/USD	DJI
Hurst index	0.4636085	0.5172341	0.3820844	0.1903221

Table 1. Estimated Hurst index for high-frequency financial time series



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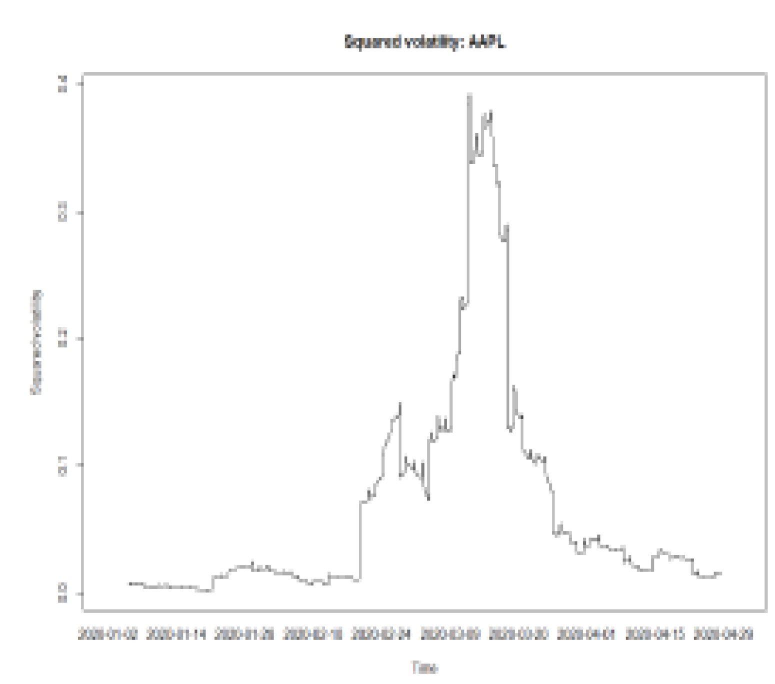

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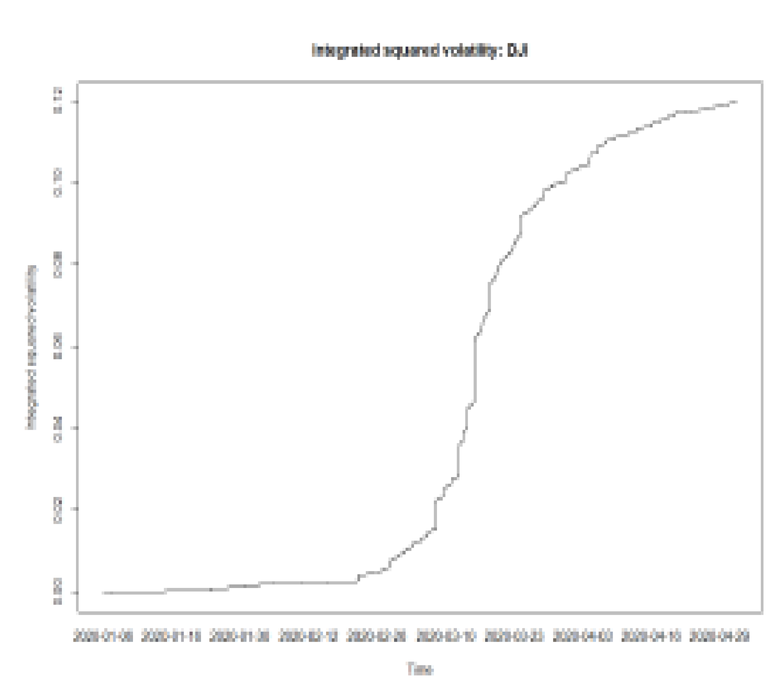

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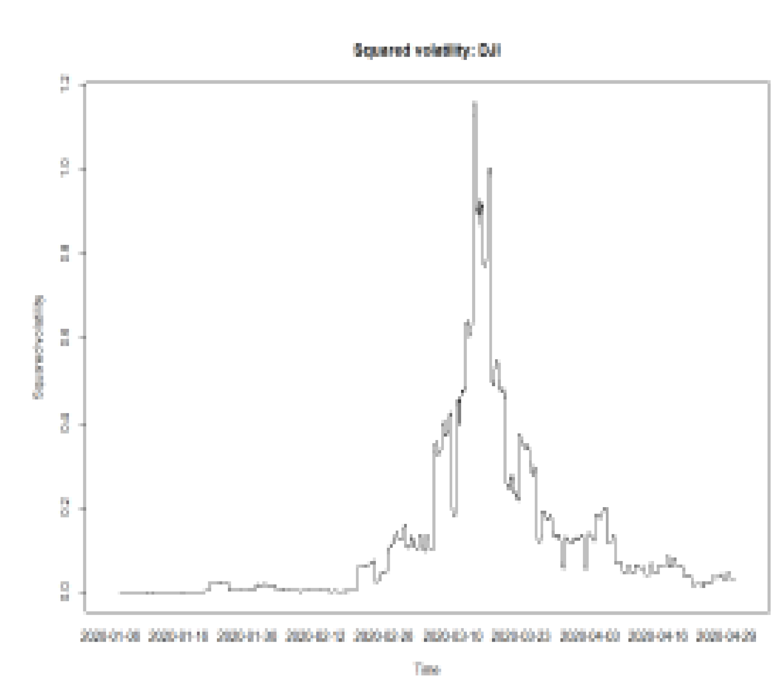

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MAIN RESULTS AND CONCLUSIONS

In this study, we have proposed a novel methodology for estimating volatility in financial time series using a fractional diffusion model. The model, which is derived from the Black-Scholes-Merton-Samuelson framework, accounts for both the market memory effect through fractional Brownian motion and the temporal changes in volatility. The Hurst index and volatility functions were estimated using the p-variations of k-order method and subsequent numerical differentiation accompanied by moving average smoothing. Our analysis, as illustrated in Figures 2-5, revealed an unprecedented surge in volatility during March 2020, which can be attributed to the onset of the COVID-19 pandemic and its impact on the global stock markets. Remarkably, the proposed model was able to accurately capture this phenomenon, highlighting its superior performance in comparison to the standard Black-Scholes-Merton-Samuelson model.

References:

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